Lecture 22: Sobolev Spaces

- Given our previous move toward weak solutions, we want to focus on spaces of functions with weak derivatives. However, it proves helpful to strengthen integrability requirements as well.

- Sobolev spaces based on L^2 are defined by

$$\mathcal{H}^m(\Omega) = \{u \in L^2(\Omega);\ D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \le m\}$$

for $m \in \mathbb{N}_0$, D^α the weak derivative. An extended family $W^{m,p}$ is defined by replacing L^2 with L^p.

$\rightsquigarrow$ Eg. $\mathcal{H}^1(\Omega)$ contains all piecewise linear functions, ~~so~~ for Ω bold, so it is a good space to use to approximate solutions by simpler functions.

- The space $\mathcal{H}^m(\Omega)$ carries an inner product

$$\langle u, v\rangle_{\mathcal{H}^m} = \sum_{|\alpha| \le m} \langle D^\alpha u, D^\alpha v\rangle$$

$\boxed{\text{Thm } 10.8}$ For $\Omega \subseteq \mathbb{R}^n$, $m \in \mathbb{N}_0$, $\mathcal{H}^m(\Omega)$ is a Hilbert space.

(We will not prove here - an exercise in convergences)

- Recall that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$ (for $f \in L^2$, there exists $\{\varphi_n\} \subset C_c^\infty$, $\{\varphi_n\} \to f$). This no longer holds in $\mathcal{H}^m$ with $m \ge 1$. In particular, we often consider

$$\mathcal{H}_0^1(\Omega) = \{u \in \mathcal{H}^m(\Omega) : \lim_{n\to\infty} \|u - \varphi_n\|_{\mathcal{H}^1} = 0 \text{ for } \varphi_n \in C_c^\infty(\Omega)\}$$

$$= \overline{C_c^\infty(\Omega)}$$

$\rightsquigarrow$ Notice $\mathcal{H}_0^1(\Omega)$ is also a Hilbert space, b/c it is closed.

- If $\partial\Omega$ is C^1, we may define a restriction to the boundary of $\mathcal{H}^1$ functions. In this case, $\mathcal{H}_0^1(\Omega)$ consists of functions whose restriction vanishes, but the true theory behind this is beyond our course (see Evans, Ch. 5).

$\boxed{10.9}$ If $u \in \mathcal{H}_0^1((a,b))$, then u is continuous on $[a,b]$ & $u(a) = u(b) = 0$.

$\boxed{Pf}$ Suppose $u \in \mathcal{H}_0^1((a,b))$ and so $\{\varphi_k\} \subseteq C_c^\infty((a,b))$ has $\lim_{k\to\infty} \|\varphi_k - u\|_{\mathcal{H}^1} = 0$

we pull a computational trick. Pick $x \in (a,b)$.

$$\varphi_j(x) - \varphi_k(x) = \int_a^x \varphi_j'(y) - \varphi_k'(y)\,dy$$

$$\leq \|\varphi_j'(y) - \varphi_k'(y)\|_{L^2} \|\chi_{[a,x]}\|_{L^2}$$

$$\leq \|\varphi_j - \varphi_k\|_{\mathcal{H}^1} \sqrt{x-a} \leq \sqrt{b-a}\,\|\varphi_j - \varphi_k\|_{\mathcal{H}^1}$$

Since $\varphi_j \to u$ in $\mathcal{H}^1$, $\{\varphi_j\}$ is cauchy in $\mathcal{H}^1$ and thus $\{\varphi_j\}$ is cauchy in L^∞ as above.

$$\|\varphi_j - \varphi_k\|_{L^\infty} \leq \sqrt{b-a}\,\|\varphi_j - \varphi_k\|_{\mathcal{H}^1}$$

Let $g = \lim_{k\to\infty} \varphi_k$ in L^∞ so $g \in C^0$ (b/c g is a uniform limit of C^∞ functions).

Since $[a,b]$ is bdd,

$$\|\varphi_j - g\|_{L^2} \leq \sqrt{(b-a)}\|\varphi_j - g\|_{L^\infty}$$

and so $\varphi_j \to g$ in L^2. We then must have $g = u$ as $\varphi_j \to u$ in L^2.

Lastly, $\varphi_j(a) \to u(a)$ gives $u(a) = 0$, and similarly $u(b) = 0$. $\square$

- This continuity of $\mathcal{H}^1$ functions doesn't generalize directly to higher dimensions. We will discuss this more shortly.

Lastly, we develop a tool for later use.

$\boxed{\text{Lemma 10.10}}$ For $\Omega \subset \tilde{\Omega} \subset \mathbb{R}^n$, the extension by 0 of an $\mathcal{H}_0^1(\Omega)$ function gives an element of $\mathcal{H}_0^1(\tilde{\Omega})$.

$\boxed{Pf}$ For $u \in \mathcal{H}_0^1(\Omega)$, let $\tilde{u}$ denote the extension-by-0 to $\tilde{\Omega}$. The weak gradient $\nabla u \in L^2(\Omega; \mathbb{R}^n)$ may also be extended by 0 to $\widetilde{\nabla u} \in L^2(\tilde{\Omega}; \mathbb{R}^n)$

• We show that $\widetilde{\nabla u}$ is the weak gradient of $\tilde{u}$.

Indeed, pick $u_k \in C_c^\infty$, $\{u_k\} \to u$ in H^1! For $\varphi \in C_c^\infty(\tilde{\Omega})$

$$\int_{\tilde{\Omega}} \varphi \nabla \tilde{u}_k dx = \int_\Omega \varphi \nabla u_k dx = -\int_\Omega u_k \nabla \varphi dx = -\int_{\tilde{\Omega}} \tilde{u}_k \nabla \varphi dx$$

$\Rightarrow$ as $k \to \infty$,

$$\int_{\tilde{\Omega}} \varphi \widetilde{\nabla u} dx = \int_\Omega \varphi \nabla u dx = -\int_\Omega u \nabla \varphi dx = -\int_{\tilde{\Omega}} \tilde{u} \nabla \varphi dx.$$

$\square$

Sobolev Regularity

Thm 10.11 Sobolev Embedding:

Suppose $U \subseteq \mathbb{R}^n$ is a bdd. domain. If $m > k + n/2$

$$H^m(U) \subseteq C^k(U).$$

⇝ The proof is a series of calculations and approximations a bit beyond our course, but it ties deeply to the Gagliardo-Nirenberg-Sobolev inequalities that focus on more general embeddings. For example, with appropriate assumptions,

$$H^m(U) \subseteq C^k(\bar{U}). \quad \text{for } k > 0.$$

⇝ Instead, we will take a route involving the connection between regularity & Fourier coefficients.

•) Set $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ to be the n-torus. We again may define, for $f \in L^2(\mathbb{T})$, $k \in \mathbb{Z}^n$:

$$c_k[f] = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ik\cdot x} f(x) dx$$

•) Arguing as we did in 1-D,

Thm 10.12 For $f \in L^2(\mathbb{T})$, $\sum_{k \in \mathbb{Z}^n} c_k[f] e^{ik\cdot x}$ converges to f in $L^2(\mathbb{T})$.

•) This directly implies $\langle f, g \rangle = (2\pi)^n \sum_{k \in \mathbb{Z}^n} c_k[f] \overline{c_k[g]}$ for $f, g \in L^2$

• Because $\mathbb{T}^n$ is periodic, we may test by functions in $C^\infty(\mathbb{T})$ instead of C_c^∞. Otherwise, $D^\alpha f$ the weak derivative of $f \in L^1_{loc}(\mathbb{T})$ is defined to be the function so

$$\int_0^{2\pi} \varphi \, D^\alpha f \, dx = (-1)^{|\alpha|} \int_0^{2\pi} f \, D^\alpha \varphi \, dx$$

for all $\varphi \in C^\infty(\mathbb{T})$.

•) Notice $D^\alpha(e^{ik\cdot x}) = (ik)^\alpha e^{ik\cdot x}$ for $(ik)^\alpha = i^{|\alpha|} k_1^{\alpha_1} \cdots k_n^{\alpha_n}$.

Then, $c_k[D^\alpha f] = (ik)^\alpha c_k[f]$ for $|\alpha| \leq m$, $f \in H^m(\mathbb{T})$.

$\boxed{\text{Th}^{\underline{m}}\ 10.13}$ A function $f \in L^2(\mathbb{T})$ lies in $H^m(\mathbb{T})$ for $m \in \mathbb{N}$ iff. $\sum_{k \in \mathbb{Z}^n} |k|^{2m} |c_k[f]|^2 < \infty$.

$\boxed{\text{Pf}}$ If $D^\alpha f \in L^2(\mathbb{T})$, $\sum_{k \in \mathbb{Z}^n} |(ik)^\alpha c_k[f]|^2 < \infty$ by Bessel's inequality.

If $f \in L^2(\mathbb{T}^n)$ has $\sum_{k \in \mathbb{Z}^n} |k|^{2m} |c_k[f]|^2 < \infty$, define for each $|\alpha| \leq m$

$$g_\alpha(x) = \sum_{k \in \mathbb{Z}^n} (ik)^\alpha c_k[f] e^{ik\cdot x}$$

has $g_\alpha \in L^2(\mathbb{T})$

and for $\varphi \in C_c^\infty(\mathbb{T}^n)$,

$$\begin{aligned} \langle g_\alpha, \varphi \rangle &= \sum (2\pi)^n (ik)^\alpha c_k[f] \overline{c_k[\varphi]} \\ &= (-1)^{|\alpha|} (2\pi)^n \sum c_k[f] \overline{c_k[D^\alpha \varphi]} \\ &= (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \end{aligned}$$

which is the same as saying $g_\alpha = D^\alpha f$. $\square$

$\boxed{\text{Thm } 10.14}$ Periodic Sobolev Embedding

If $m > q + n/2$,

$$\mathcal{H}^m(\mathbb{T}^n) \subset C^q(\mathbb{T}^n).$$

$\boxed{\text{Pf}}$ Recall $\ell^2(\mathbb{Z}^n)$, the Hilbert space of functions $\mathbb{Z}^n \to \mathbb{C}$ with inner product $\langle \beta, \gamma \rangle = \sum_{k \in \mathbb{Z}^n} \beta(k)\overline{\gamma(k)}$

Consider $\beta(k) = (1+|k|)^{-m}$ where

$$\|\beta\|_{\ell^2}^2 = \sum_{k \in \mathbb{Z}^n} (1+|k|)^{-2m} \leq \int_{\mathbb{R}^n} (1+|x|)^{-2m} dx$$
$$= A_n \int_0^\infty (1+r)^{-2m} r^{n-1} dr$$

which is finite if $2m > n$ (in which case $\beta \in \ell^2(\mathbb{Z})$).

Let $f \in \mathcal{H}^m(\mathbb{T}^n)$ for $m > \cancel{q+}n/2$. Define $\gamma(k) = (1+|k|)^m |c_k[f]|$ which is in $\ell^2(\mathbb{Z}^n)$ by the prev. thm.

and $\langle \beta, \gamma \rangle_{\ell^2} = \sum_{k \in \mathbb{Z}^n} |c_k[f]| \leq \|\beta\|_{\ell^2} \|\gamma\|_{\ell^2} < \infty$

Such that $\sum_{k \in \mathbb{Z}^n} c_k[f] e^{ikx} \to f$ uniformly and $f \in C^0(\mathbb{T})$.

To apply to higher derivatives, $f \in \mathcal{H}^m(\mathbb{T}^n)$ for $m > q + n/2$ means that for $|\alpha| \leq q$, $D^\alpha f \in \mathcal{H}^{m-|\alpha|}(\mathbb{T}^n) \subseteq C^0(\mathbb{T})$. $\square$

Finally, we prove thm 10.11

Suppose $u \in \mathcal{H}^m(U)$, $U \subseteq \mathbb{R}^n$. Let $x_0 \in U$, so for some small $\varepsilon > 0$, $B(x_0, \varepsilon) \subset \Omega$.

Find $\psi \in C_c^\infty(B(x_0, \varepsilon))$ so $\psi \equiv 1$ on $B(x_0, \varepsilon/2)$. Assuming $\varepsilon < 2\pi$, ~~the~~ $u\psi \in \mathcal{H}^m(U)$ may be extended to a function in $\mathcal{H}^m(\mathbb{T}^n)$ by periodicity.

Since $u = u\psi$ in $B(x_0, \varepsilon/2)$, the prev. thm. shows $u \in C^k(B(x_0, \varepsilon/2))$ for $m > k + n/2$. Repeat for each $x_0 \in U$.

$\square$